Complements to the paper "$W^{1,1}$-maps with values into S^1"

Haïm Brezis, Petru Mironescu, Augusto C. Ponce

COMPLEMENTS TO THE PAPER "$W^{1,1}$-MAPS WITH VALUES INTO S^1" 07/05/04

Haïm Brezis[(1),(2)], Petru Mironescu[(3)] and Augusto C. Ponce[(1),(2)]

The purpose of these notes is to complement some of our results in [BMP]. We also establish some of the claims we stated there without proof.

A. Extending Theorem 10 to other seminorms in $W^{1,1}(\Omega; S^1)$.

In view of Theorem 10, it is natural to introduce the following quantity

$$\rho(P, N) = \frac{1}{2\pi} \operatorname{Inf} \Big\{ [g]_{W^{1,1}} \ ; \ g \in W^{1,1}(\Omega; S^1), \ T(g) = 2\pi(\delta_P - \delta_N) \Big\}.$$

Here, $[\ ]_{W^{1,1}}$ is a general given semi-norm on $W^{1,1}(\Omega; \mathbb{R}^2)$ equivalent to $|\ |_{W^{1,1}}$. We require from $[\ ]_{W^{1,1}}$ some structural properties :

(P1) $[\alpha g]_{W^{1,1}} = [g]_{W^{1,1}}, \ \forall\, g \in W^{1,1}(\Omega; \mathbb{R}^2), \ \forall\, \alpha \in S^1$;

(P2) $[\bar{g}]_{W^{1,1}} = [g]_{W^{1,1}}, \ \forall\, g \in W^{1,1}(\Omega; \mathbb{R}^2)$;

(P3) $[gh]_{W^{1,1}} \le \|g\|_{L^\infty} [h]_{W^{1,1}} + \|h\|_{L^\infty} [g]_{W^{1,1}}, \ \forall\, g, h \in W^{1,1}(\Omega; \mathbb{R}^2) \cap L^\infty$.

It follows from (P3) that ρ is a distance.

[We refer the reader to [Mi] for the solution of the exercise given at the end of Section 3.]

Alternatively, we may define ρ starting from maps in $\mathcal{R}$:

Lemma A1. *We have*

$$\rho(P, N) = \frac{1}{2\pi} \operatorname{Inf} \left\{ [g]_{W^{1,1}} \ \middle| \ \begin{array}{c} g \in C^\infty(\Omega \setminus \{P, N\}; S^1) \cap W^{1,1}, \\ \deg(g, P) = +1, \ \deg(g, N) = -1 \end{array} \right\}.$$

Proof. It suffices to prove that, for $g \in W^{1,1}(\Omega; S^1)$ such that $T(g) = 2\pi(\delta_P - \delta_N)$, we may find a sequence $(g_n) \subset \mathcal{R}$ such that $T(g_n) = 2\pi(\delta_P - \delta_N)$ and $g_n \to g$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX in $W^{1,1}$. Note that the desired conclusion is invariant with respect to orientation-preserving diffeomorphisms of Ω. Therefore, we may assume that $\Omega = S^2$ and that P, N are respectively the North and the South pole of S^2. Let $h(x,y,z) = \dfrac{(x,y)}{|(x,y)|}$. Then $h \in \mathcal{R}$, $T(h) = 2\pi(\delta_P - \delta_N)$. It follows that $k = g\bar{h} \in Y$. Thus, we may find a sequence $(k_n) \subset C^\infty(\Omega; S^1)$ such that $k_n \to k$ in $W^{1,1}$. Set $g_n = hk_n$. Then $g_n \in \mathcal{R}$, $T(g_n) = 2\pi(\delta_P - \delta_N)$ and $g_n \to g$ in $W^{1,1}$.

Another simple property of ρ is

$$C_1 d(P,N) \le \rho(P,N) \le C_2 d(P,N) \tag{A1}$$

for some $C_1, C_2 > 0$ independent of P, N. This follows from the equivalence of $|\ |_{W^{1,1}}$ and $[\ ]_{W^{1,1}}$.

Part of Theorem 10 holds in this more general setting :

Theorem A1. *Let $P_i, N_i \in \Omega$ be such that*

$$\sum \rho(P_i, N_i) < \infty \quad \textit{(or, equivalently, } \sum d(P_i, N_i) < \infty\textit{).}$$

Set

$$\widetilde{L} = \operatorname{Inf}\Big\{\sum \rho(\widetilde{P}_j, \widetilde{N}_j)\ ;\ \sum(\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j}) = \sum(\delta_{P_i} - \delta_{N_i})\Big\}.$$

Then

$$\frac{1}{2\pi} \operatorname{Inf}\Big\{[g]_{W^{1,1}}\ ;\ g \in W^{1,1}(\Omega; S^1),\ T(g) = 2\pi \sum(\delta_{P_i} - \delta_{N_i})\Big\} \le \widetilde{L}.$$

Proof. Let $\varepsilon > 0$ and let $\widetilde{P}_j, \widetilde{N}_j$ be such that

$$\sum(\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j}) = \sum(\delta_{P_i} - \delta_{N_i}) \quad \text{and} \quad \sum \rho(\widetilde{P}_j, \widetilde{N}_j) \le \widetilde{L} + \varepsilon.$$

By definition, for each j we may find some $g_j \in W^{1,1}(\Omega; S^1)$ such that

$$T(g_j) = 2\pi(\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j})$$

and

$$[g_j]_{W^{1,1}} \le 2\pi\rho(\widetilde{P}_j, \widetilde{N}_j) + \frac{\varepsilon}{2^j}.$$

We claim that there is a sequence $k_n \to \infty$ such that $\Pi_{j=1}^{k_n} g_j \xrightarrow{n} g$ in $W^{1,1}$ for some $g \in W^{1,1}(\Omega; S^1)$. By Lemma 1, this implies that

$$T(g) = 2\pi \sum_j (\delta_{\widetilde{P}_j} - \delta_{\widetilde{N}_j}) = 2\pi \sum_i (\delta_{P_i} - \delta_{N_i}).$$

Using (P3), we will also have $[g]_{W^{1,1}} \leq 2\pi\widetilde{L} + \varepsilon$. Therefore, the conclusion of Theorem A1 follows if we prove the existence of the sequence (k_n). We adapt below an argument used in [BBM2]. Set $H = \sum_{j\geq 1} |\nabla g_j| \in L^1$ (this uses the equivalence of $|\ |_{W^{1,1}}$ and $[\ ]_{W^{1,1}}$). Since $\left|\nabla\left(\Pi_{j=1}^k g_j\right)\right| \leq H$, $\forall k$, we may find a sequence $k_n \to \infty$ and a map $g \in BV(\Omega; S^1)$ such that $h_n = \Pi_{j=1}^{k_n} g_j \to g$ a.e. Then, for $m > n$, we have

$$\begin{aligned}
|h_m - h_n|_{W^{1,1}} &= |h_n(h_m\bar{h}_n - 1)|_{W^{1,1}} \\
&\leq |h_m\bar{h}_n|_{W^{1,1}} + \|(1 - h_m\bar{h}_n)\nabla h_n\|_{L^1} \\
&\leq \sum_{j=k_n+1}^{k_m} \|\nabla g_j\|_{L^1} + \|(1 - h_m\bar{h}_n)H\|_{L^1} = A_{m,n} + B_{m,n}.
\end{aligned}$$

Let $0 < \delta < 1$. Then, clearly, $A_{m,n} < \delta$ provided m, n are sufficiently large. On the other hand,

$$B_{m,n} \leq \delta\|H\|_{L^1} + 2\int_{\{x:|1-h_m(x)\bar{h}_n(x)|\geq\delta\}} |H|.$$

Note that

$$\big\{x; |1 - h_m(x)\bar{h}_n(x)| \geq \delta\big\} \subset \Big\{x; |g(x) - h_m(x)| \geq \frac{\delta}{3}\Big\} \cup \Big\{x; |g(x) - h_n(x)| \geq \frac{\delta}{3}\Big\}.$$

Since $h_n \to g$ a.e., we find that $B_{m,n} \leq \delta(\|H\|_{L^1} + 1)$, provided m, n are sufficiently large. Therefore, (h_n) is a Cauchy sequence in $W^{1,1}$ and converges to the above g in $W^{1,1}$.

It is not clear whether the reverse inequality in Theorem A1 is valid in general :

Open Problem 8. Let $P_i, N_i \in \Omega$ be such that $T(g) = 2\pi\sum_i(\delta_{P_i} - \delta_{N_i})$. Is it true that

$$[g]_{W^{1,1}} \geq 2\pi\widetilde{L}\ ?$$

Note that, by definition, the answer is yes if $T(g) = 2\pi(\delta_P - \delta_N)$.

B. Proof of Theorems 1′, 3′, and 5′.

Proof of Theorem 3′. Let us first assume that $g \in C^\infty(\overline{\Omega}\backslash\{a_1, \dots, a_k\}; S^1) \cap W^{1,1}$. It is then easy to see that

$$\langle T(g), \zeta\rangle = 2\pi\sum_{j=1}^k d_j\zeta(a_j) + \int_{\partial\Omega} (g \wedge g_\tau)\zeta, \quad \forall\zeta \in \mathrm{Lip}\,(\Omega; \mathbb{R}),$$

where d_j denotes the topological degree of g with respect to any small circle centered at a_j. In particular,

$$\langle T(g), \zeta\rangle = 2\pi \sum_{j=1}^{k} d_j \zeta(a_j), \quad \forall \zeta \in W_0^{1,\infty}(\Omega). \tag{B1}$$

Note that, in general, $\sum_j d_j \neq 0$. This means that we do not have necessarily the same number of positive and negative points as before. In order to compensate this, we insert points from $\partial\Omega$ into (B1). Since $\zeta = 0$ on $\partial\Omega$, equality in (B1) remains true. We can then relabel the points a_j as $P_1, \dots, P_\ell$, $N_1, \dots, N_\ell$, according to their multiplicity d_j, so that (B1) becomes

$$T(g) = 2\pi \sum_{j=1}^{\ell} (\delta_{P_j} - \delta_{N_j}) \quad \text{in } W_0^{1,\infty}(\Omega).$$

For a general $g \in W^{1,1}(\Omega; S^1)$, we argue by density using Lemma 2 to conclude that

$$T(g) = 2\pi \sum (\delta_{P_i} - \delta_{N_i}) \quad \text{in } \big[W_0^{1,\infty}(\Omega)\big]^*.$$

Note that d_Ω induces a metric on the space $\overline{\Omega}/\partial\Omega$, where $\partial\Omega$ is identified with a single point. Moreover, Lipschitz functions ζ on $\overline{\Omega}/\partial\Omega$ with $|\zeta|_{\mathrm{Lip}} \leq 1$ and $\zeta(\partial\Omega) = 0$ correspond to elements in $W_0^{1,\infty}(\Omega)$ such that $\|\nabla\zeta\|_{L^\infty} \leq 1$. Applying Lemma 12$'$ in [BBM2] to $\overline{\Omega}/\partial\Omega$, we obtain

$$L(g) = \operatorname{Inf} \sum_i d_\Omega(P_i, N_i).$$

Remark B1. The main new feature when Ω is a bounded domain in $\mathbb{R}^2$ is that a minimal connection is made of segments from a positive singularity P_i to some negative N_j, but we can also have line segments joining the singularities P_i, N_i to the boundary $\partial\Omega$. This is the analog of Example 3 in [BCL].

Proof of Theorem 1$'$. The proof of $E(g) = E_{\mathrm{rel}}(g)$ is exactly the same as in Proposition 2 and we shall omit it.

We are left to show that

$$E(g) = \int_\Omega |\nabla g| + 2\pi L(g). \tag{B2}$$

Let $\varphi \in BV(\Omega; \mathbb{R})$ be such that $g = \mathrm{e}^{i\varphi}$. Using Vol'pert's chain rule as in the proof of Lemma 5, we have

$$|D\varphi|_{\mathcal{M}(\Omega)} = |g|_{W^{1,1}} + |g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)}. \tag{B3}$$

We claim that

$$|g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} \geq 2\pi L(g). \tag{B4}$$

In fact, for every $\zeta \in C_0^\infty(\Omega)$ such that $\|\nabla\zeta\|_{L^\infty} \leq 1$,

$$|g \wedge \nabla g - D\varphi|_{\mathcal{M}(\Omega)} \geq \int_\Omega (g \wedge \nabla g) \cdot \nabla^\perp \zeta - \int_\Omega D\varphi \cdot \nabla^\perp \zeta = \langle T(g), \zeta \rangle.$$

Taking the supremum with respect to ζ, we conclude that (B4) holds.

Inequality $\geq$ in (B2) follows immediately from (B3) and (B4).

We now establish $\leq$ in (B2). Let us assume for the moment that g is smooth outside finitely many points $a_1, \ldots, a_k$, and that g has topological degree ± 1 at each one of those points. Let $\mathcal{C}$ be a minimal connection between those points with respect to the distance d_Ω.

Note that on any closed curve contained in $\Omega \backslash \mathcal{C}$, g has zero topological degree. We conclude that g has a smooth lifting φ on $\Omega \backslash \mathcal{C}$. Moreover, as we cross any one of the line segments of $\mathcal{C}$, φ jumps by 2π. Thus, $\varphi \in BV(\Omega; \mathbb{R})$ and

$$\int_\Omega |D\varphi| = \int_\Omega |\nabla g| + 2\pi|\mathcal{C}| = \int_\Omega |\nabla g| + 2\pi L(g).$$

We can now argue by density, using Lemma 2, to conclude that for any $g \in W^{1,1}(\Omega; S^1)$ there exists $\varphi \in BV(\Omega; \mathbb{R})$ such that $g = \mathrm{e}^{i\varphi}$ a.e. in Ω and

$$\int_\Omega |D\varphi| \leq \int_\Omega |\nabla g| + 2\pi L(g). \tag{B5}$$

This concludes the proof of the theorem.

Proof of Theorem 5′. Using exactly the same argument as in the proof of Proposition 3, we have

$$|\langle \mathrm{Det}\,(\nabla g), \zeta \rangle| \leq TV(g) \|\zeta\|_{L^\infty}, \quad \forall \zeta \in C_0^\infty(\Omega). \tag{B6}$$

Thus, if $TV(g) < \infty$, then $\mathrm{Det}\,(\nabla g) \in \mathcal{M}(\Omega)$. We now apply Proposition 3.2 in [S] (see also [P]) to the quotient space $\overline{\Omega}/\partial\Omega$. We conclude that there exist distinct points $a_1, \ldots, a_k \in \Omega$ and nonzero integers $d_1, \ldots, d_k$ such that

$$\mathrm{Det}\,(\nabla g) = \pi \sum_{j=1}^k d_j \delta_{a_j}. \tag{B7}$$

We now define

$$h(x) := \left(\frac{x-a_1}{|x-a_1|}\right)^{-d_1} \cdots \left(\frac{x-a_k}{|x-a_k|}\right)^{-d_k} g(x) \quad \text{for a.e. } x \in \Omega.$$

Clearly, $\operatorname{Det}(\nabla h) = 0$ in $\mathcal{D}'(\Omega)$. It follows from the analog of Theorem 7 for domains in $\mathbb{R}^2$ (see also [D]) that h has a lifting in $W^{1,1}$. In other words, we can find $\varphi \in W^{1,1}(\Omega;\mathbb{R})$ such that $h = \mathrm{e}^{i\varphi}$ a.e. in Ω. We then conclude that

$$g(x) = \left(\frac{x-a_1}{|x-a_1|}\right)^{d_1} \cdots \left(\frac{x-a_k}{|x-a_k|}\right)^{d_k} \mathrm{e}^{i\varphi(x)} \quad \text{for a.e. } x \in \Omega.$$

Arguing as in the proof of Theorem 5, this implies that

$$TV(g) \leq \pi \sum_{j=1}^{k} |d_j|. \tag{B8}$$

The reverse inequality already follows from (B6). We then conclude that (6.8) holds.

Conversely, if $\operatorname{Det}(\nabla g) \in \mathcal{M}(\Omega)$, then (B7) holds. The above argument then shows that $TV(g) < \infty$ and

$$TV(g) = |\operatorname{Det}(\nabla g)|_{\mathcal{M}} = \pi \sum_{i=1}^{k} |d_i|.$$

C. Proof of Theorem 3''''.

Theorem 3'''' follows immediately from Theorem 3 and the next

Lemma C1. *Given $g \in W^{1,1}(S^2;\Gamma)$, we define*

$$g_j := \frac{g-a_j}{|g-a_j|} \in W^{1,1}(S^2;S^1).$$

Then,

$$\operatorname{Det}(\nabla g) = \frac{1}{\pi} \sum_j |A_j| \operatorname{Det}(\nabla g_j) \quad \textit{in } \mathcal{D}'(S^2). \tag{C1}$$

The proof of Lemma C1 relies on the following

Lemma C2. *For any $u \in W^{1,1}(S^1;\Gamma)$, we have*

$$\frac{1}{2}\int_{S^1} u \wedge u_\tau = \sum_j |A_j| \deg \frac{u-a_j}{|u-a_j|}. \tag{C2}$$

Proof of Lemma C2.

Step 1. Γ is a simple curve.
It is well-known that (C2) holds if $u \in C^1(S^1;\Gamma)$ (see, e.g., [N]). By approximation, we conclude that (C2) is also true for any $u \in W^{1,1}(S^1;\Gamma)$.

Step 2. Assume Γ has finitely many self-intersections, say $q_1, \dots, q_k$.
Since u is continuous, the set

$$S^1 \backslash u^{-1}(\{q_1, \dots, q_k\})$$

is open and can be written as a countable union of open arcs in S^1. Let α_1 be such an arc. It is easy to see that we can select disjoint arcs $\alpha_2, \dots, \alpha_j$ (oriented anticlockwise) such that u at the positive endpoint of α_i coincides with the value of u at the negative endpoint of α_{i+1} for $i = 1, \dots, j$, with the convention that $\alpha_{j+1} = \alpha_1$. By removing arcs from this list if necessary, we can assume that each point q_i appears only twice in the list

$$\big\{u(\partial\alpha_1), \dots, u(\partial\alpha_j)\big\}.$$

This construction induces a function $\tilde{u} \in W^{1,1}(S^1;\Gamma)$ such that

(*a*) $\tilde{u} = u$ on $\alpha_1 \cup \dots \cup \alpha_j$;

(*b*) $\tilde{u}$ is locally constant on $S^1 \backslash \alpha_1 \cup \dots \cup \alpha_j$.

By construction, $\tilde{u}(S^1)$ is a subset of a Jordan curve $\tilde{\Gamma}$ contained in Γ. Let $A_{t_1}, \dots, A_{t_\ell}$ be the components of $\mathbb{R}^2 \backslash \Gamma$ enclosed by $\tilde{\Gamma}$. By our first step, we have

$$\frac{1}{2}\int_{S^1} \tilde{u} \wedge \tilde{u}_\tau = \big|A_{t_1} \cup \dots \cup A_{t_\ell}\big| \deg \frac{\tilde{u}-a}{|\tilde{u}-a|}. \tag{C3}$$

for some point a inside $\tilde{\Gamma}$. Note, however, that

$$\deg \frac{\tilde{u}-a}{|\tilde{u}-a|} = \deg \frac{\tilde{u}-a_{t_s}}{|\tilde{u}-a_{t_s}|}, \quad \forall s = 1, \dots, \ell \ ;$$

moreover,

$$\deg \frac{\tilde{u}-a_j}{|\tilde{u}-a_j|} = 0, \quad \text{if } j \notin \{t_1, \dots, t_\ell\}.$$

We can thus rewrite (C3) as

$$\text{(C4)} \qquad \frac{1}{2}\int_{S^1} \tilde{u} \wedge \tilde{u}_\tau = \sum_{s=1}^{\ell} |A_{t_s}| \deg \frac{\tilde{u} - a_{t_s}}{|\tilde{u} - a_{t_s}|} = \sum_j |A_j| \deg \frac{\tilde{u} - a_j}{|\tilde{u} - a_j|}.$$

We can proceed with the construction of $\tilde{u}$ and "decompose" $u \in W^{1,1}(S^1;\Gamma)$ as $\tilde{u}_1, \tilde{u}_2, \ldots$ so that

(*a*) $\tilde{u}_i \in W^{1,1}(S^1;\Gamma)$ and $\tilde{u}_i(S^1)$ is contained in some Jordan curve for every i ;

(*b*) $u_\tau = \tilde{u}_{1\tau} + \tilde{u}_{2\tau} + \cdots$ in S^1 ;

(*c*) $\tilde{u}_i$ coincides with u on finitely many arcs in $S^1 \backslash u^{-1}(\{q_1, \ldots, q_k\})$ and $\tilde{u}_i$ is locally constant outside those arcs.

By (C4), we have

$$\frac{1}{2}\int_{S^1} \tilde{u}_i \wedge \tilde{u}_{i\tau} = \sum_j |A_j| \deg \frac{\tilde{u}_i - a_j}{|\tilde{u}_i - a_j|}, \quad \forall i.$$

Note that, by (*b*) and (*c*),

$$\int_{S^1} u \wedge u_\tau = \sum_i \int_{S^1} \tilde{u}_i \wedge \tilde{u}_{i\tau}.$$

For the same reason,

$$\deg \frac{u - a_j}{|u - a_j|} = \sum_i \deg \frac{\tilde{u}_i - a_j}{|\tilde{u}_i - a_j|}, \quad \forall j.$$

We conclude that (C2) holds.

Proof of Lemma C1. Let $g \in W^{1,1}(S^2;\Gamma)$. By the coarea formula (see [BBM2]), we have

$$\text{(C5)} \qquad \langle \operatorname{Det}(\nabla g), \zeta \rangle = \int_{\mathbb{R}} \left(\int_{\Sigma_\lambda} g \wedge g_\tau \right) d\lambda,$$

where $\zeta \in C^\infty(S^2)$, and $\Sigma_\lambda = \big\{x \in S^2 \ ; \ \zeta(x) = \lambda\big\}$ is equipped with the appropriate orientation, whenever λ is a regular value of ζ.

Recall that, for a.e. $\lambda \in \mathbb{R}$, $g|_{\Sigma_\lambda}$ belongs to $W^{1,1}$. Applying Lemma C2 to $g|_{\Sigma_\lambda}$ for such λs we get

$$\int_{\Sigma_\lambda} g \wedge g_\tau = 2\sum_j |A_j| \deg g_j = \frac{1}{\pi}\sum_j |A_j| \int_{\Sigma_\lambda} g_j \wedge g_{j\tau}.$$

Integrate both sides of the identity above with respect to λ. Using (C5), we conclude that

$$\langle \mathrm{Det}\,(\nabla g), \zeta\rangle = \frac{1}{\pi}\sum_j |A_j|\langle \mathrm{Det}\,(\nabla g_j), \zeta\rangle, \quad \forall \zeta \in C^\infty(S^2).$$

This establishes (C1).

We also call the attention of the reader to the following analog of Lemma 12′ in [BBM2] :

Proposition C1. *Let X be a metric space. Given two sequences (P_i), (N_i) in X and nonnegative numbers α_i such that $\sum_i \alpha_i d(P_i, N_i) < \infty$, let*

$$T = \sum_i \alpha_i(\delta_{P_i} - \delta_{N_i}) \quad \textit{in } \big[\mathrm{Lip}(X)\big]^*. \tag{C6}$$

Define

$$L = \operatorname*{Sup}_{\substack{\zeta\in \mathrm{Lip}(X)\\ |\zeta|_{\mathrm{Lip}}\le 1}} \langle T, \zeta\rangle.$$

Then,

$$L = \mathrm{Inf}\sum_i \alpha_i d(P_i, N_i), \tag{C7}$$

where the infimum is taken over all sequences (P_i), (N_i) in X and numbers $\alpha_i \ge 0$ such that (C6) *holds.*

Proof. Let us denote by $\tilde L$ the infimum in (C7). Clearly, $L \le \tilde L$. We now establish the reverse inequality.

Let $\varepsilon > 0$. We take $k \ge 1$ sufficiently large so that

$$\sum_{i>k} \alpha_i d(P_i, N_i) < \varepsilon.$$

Without loss of generality, we can assume that each α_i is rational for $i = 1, \ldots, k$. We then choose an integer $J \ge 1$ sufficiently large so that $J\alpha_i$ is an integer for every $i = 1, \ldots, k$. Write the points P_i, N_i as $p_1, p_2, \ldots$ and $n_1, n_2, \ldots$, with multiplicity $J\alpha_i$. It follows from Lemma 4.2 in [BCL] that we can find $\zeta_0 \in \mathrm{Lip}(X)$, with $|\zeta_0|_{\mathrm{Lip}} \le 1$, such that, after relabeling the points n_j if necessary, we have :

$$\sum_{i=1}^k J\alpha_i\big[\zeta_0(P_i) - \zeta_0(N_i)\big] = \sum_j \big[\zeta_0(p_j) - \zeta_0(n_j)\big] = \sum_j d(p_j, n_j).$$

Thus,

$$\begin{aligned} L &\geq \sum_{i=1}^{\infty} \alpha_i \big[\zeta_0(P_i) - \zeta_0(N_i)\big] \\ &\geq \sum_{i=1}^{k} \alpha_i \big[\zeta_0(P_i) - \zeta_0(N_i)\big] - \varepsilon \\ &= \frac{1}{J} \sum_j d(p_j, n_j) - \varepsilon \geq \sum_j \frac{1}{J} d(p_j, n_j) + \sum_{i>k} \alpha_i d(P_i, N_i) - 2\varepsilon. \end{aligned}$$

Note that

$$T = \sum_j \frac{1}{J} (\delta_{p_j} - \delta_{n_j}) + \sum_{i>k} \alpha_i (\delta_{P_i} - \delta_{N_i}).$$

We conclude that $L \geq \tilde{L} - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this implies $L \geq \tilde{L}$. Thus, $L = \tilde{L}$ as claimed.

D. Proof of Theorems 1″ and 2″.

We begin with a few preliminary results

Lemma D1. *Given $\varepsilon > 0$, let $\Phi_\varepsilon : \bar{B}^N \to \bar{B}^N$ be defined as*

$$\Phi_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| \leq \varepsilon, \\ \dfrac{|x| - \varepsilon}{1 - \varepsilon} \dfrac{x}{|x|} & \text{if } \varepsilon < |x| \leq 1. \end{cases}$$

Then, for every $f \in C^\infty(\bar{B}^N; \mathbb{R}^M)$, we have

$$f \circ \Phi_\varepsilon \to f \quad \text{in } W^{1,N}(B^N).$$

Proof. Given $x \in B^N$, $|x| \geq \varepsilon$, let $r = |x|$ and $\psi_\varepsilon(r) = \frac{r-\varepsilon}{(1-\varepsilon)r}$. Using this notation, we have

$$\Phi'_\varepsilon(x) = \psi_\varepsilon(r) \, \mathrm{Id} + \Big(\frac{x_i x_j}{r} \psi'_\varepsilon(r) \Big).$$

Since $\psi'_\varepsilon(r) = \frac{\varepsilon}{(1-\varepsilon)r^2}$,

$$\Big| \frac{x_i x_j}{r} \psi'_\varepsilon(r) \Big| \leq \frac{C\varepsilon}{r}.$$

Moreover,

$$|\Phi_\varepsilon(x) - x| \leq C\varepsilon \quad \text{and} \quad |\psi_\varepsilon(r) - 1| \leq \frac{C\varepsilon}{r}.$$

We then have

$$\begin{aligned}\big|\nabla f_\varepsilon(x) - \nabla f(x)\big| &= \big|\,{}^t\Phi'_\varepsilon(x)\,\nabla f(\Phi_\varepsilon(x)) - \nabla f(x)\big| \\ &\leq \big|\nabla f(\Phi_\varepsilon(x)) - \nabla f(x)\big| + \big|\nabla f(\Phi_\varepsilon(x))\big|\,\big|\operatorname{Id} - {}^t\Phi'_\varepsilon(x)\big| \\ &\leq C|\Phi_\varepsilon(x) - x| + C|\psi_\varepsilon(r) - 1| + \frac{C\varepsilon}{r} \leq \frac{C\varepsilon}{r}\end{aligned}$$

for $|x| \geq \varepsilon$. Therefore,

$$\int_{B^N} |\nabla f_\varepsilon - \nabla f|^N \leq C\varepsilon^N \int\limits_{\varepsilon \leq |x| \leq 1} \frac{dx}{|x|^N} + \int\limits_{|x|<\varepsilon} |\nabla f| \to 0 \quad \text{as } \varepsilon \to 0.$$

Next, we establish the following

Lemma D2. *Given $f \in C^\infty(\bar{B}^N \times [0,1]; \mathbb{R}^M)$, let*

$$f_\varepsilon(x,t) = f(\Phi_\varepsilon(x), t).$$

Then

$$f_\varepsilon \to f \quad \text{in } W^{1,N}(B^N \times [0,1]).$$

Proof. Note that

$$\frac{\partial f_\varepsilon}{\partial t}(x,t) - \frac{\partial f}{\partial t}(x,t) = \begin{cases} \dfrac{\partial f}{\partial t}(0,t) - \dfrac{\partial f}{\partial t}(x,t) & \text{if } |x| < \varepsilon, \\ \dfrac{\partial f}{\partial t}(\Phi_\varepsilon(x),t) - \dfrac{\partial f}{\partial t}(x,t) & \text{if } |x| \geq \varepsilon. \end{cases}$$

Thus, since $|\Phi_\varepsilon(x) - x| \leq C\varepsilon$, we have

$$\Big|\frac{\partial f_\varepsilon}{\partial t} - \frac{\partial f}{\partial t}\Big| \leq C\varepsilon.$$

The result now immediately follows from Lemma D1.

As a consequence of Lemma D2 above, any map $f \in C^\infty(\bar{B}^N \times [0,1]; S^k)$ can be approximated in $W^{1,N}$ by maps f_ε such that

$$f_\varepsilon = f \quad \text{on } S^{N-1} \times [0,1] \quad \text{and} \quad f_\varepsilon(x,t) = g_\varepsilon(t) \quad \text{if } |x| \leq \varepsilon.$$

Lemma D3. *Given $g \in C^\infty([0,1];\mathbb{R}^M)$, let*

$$f(x,t) = g(t) \quad \forall (x,t) \in \bar{B}^N \times [0,1].$$

Then, there exists a sequence (f_ε) in $W^{1,N}$ such that

$$f_\varepsilon(x,t) = const \quad \textit{if } |x| \le \varepsilon, \quad f_\varepsilon(x,t) = g(t) \quad \textit{if } |x| = 1,$$

and

$$f_\varepsilon \to f \quad \textit{in } W^{1,N}(B^N \times [0,1]).$$

Proof. Let $\zeta_\varepsilon : \bar{B}^N \to \mathbb{R}$ be given by

$$\zeta_\varepsilon(x) = \begin{cases} \dfrac{\log \frac{|x|}{\varepsilon}}{\log \frac{1}{\varepsilon}} & \text{if } \varepsilon \le |x| \le 1, \\ 0 & \text{if } |x| < \varepsilon. \end{cases}$$

It is easy to see that

$$\int_{B^N} |\nabla \zeta_\varepsilon|^N \to 0 \quad \text{and} \quad \int_{B^N} |\zeta_\varepsilon - 1|^N \to 0.$$

Since $N \ge 2$, there exists $Q \in S^N$ such that $Q \notin g([0,1])$. Let $\Psi : S^N \backslash \{Q\} \to \mathbb{R}^N$ denote the stereographic projection. Set $F = \Psi \circ f$. Clearly, in order to establish the lemma, it suffices to approximate F in $W^{1,N}$ by a sequence F_ε such that $|F_\varepsilon| \le C$,

$$F_\varepsilon(x,t) = \text{const} \quad \text{if } |x| \le \varepsilon \quad \text{and} \quad F_\varepsilon(x,t) = F(x,t) \quad \text{if } |x| = 1.$$

Set $G(t) = \Psi \circ g(t)$ and $F_\varepsilon(x,t) = \zeta_\varepsilon(x) G(t)$. We then have

$$\iint |\nabla_x F_\varepsilon - \nabla F|^N \, dx \, dt = \int |\nabla \zeta_\varepsilon|^N \int |G|^N \to 0$$

and

$$\iint |\partial_t F_\varepsilon - \partial_t F|^N \, dx \, dt = \int |1 - \zeta_\varepsilon|^N \int |G'|^N \to 0.$$

The proof of Lemma D3 is complete.

We conclude from Lemma D3 that, given any $u \in W^{1,N-1}(S^N; S^{N-1})$, $N \ge 3$, there exists a sequence (u_n) such that $u_n \to u$ in $W^{1,N}$, where each u_n satisfies the following properties :

(i) u_n has a finite number of point singularities P_i, N_i ;

(ii) u_n is homogeneous of degree 0 in a neighborhood of each singularity ;

(iii) $u_n \equiv Q$ in some conic neighborhood of a geodesic joining P_i and N_i.

In fact, (ii) holds since every topological singularity may be approximated by homogeneous singularities (see Lemma E.5 below). We then apply Lemmas D.2 and D.3 to obtain property (iii).

We shall say that a map is *good* if it satisfies properties (i)–(iii).

Proof of Theorem 2″. Our goal is to show that, for every $u \in W^{1,N-1}(S^N; S^{N-1})$,

$$\text{(D1)} \qquad \operatorname*{Inf}_{v \in C^\infty(S^N; S^{N-1})} \int_{S^N} |D(u) - D(v)| = \sigma_N L(u).$$

Proof of “$\geq$”. For every $\zeta \in \operatorname{Lip}(S^N)$ such that $\|\nabla\zeta\|_{L^\infty} \leq 1$, we have

$$\int_{S^N} |D(u) - D(v)| \geq \int_{S^N} \big[D(u) - D(v)\big] \cdot \nabla\zeta = \int_{S^N} D(u) \cdot \nabla\zeta = \langle T(u), \zeta\rangle.$$

Taking the supremum with respect to ζ, we get

$$\int_{S^N} |D(u) - D(v)| \geq \sigma_N L(u) \quad \forall v \in C^\infty(S^N; S^{N-1}).$$

Proof of “$\leq$”. Assume u is a good map. We shall assume for simplicity that u has a single dipole P, N. Given $\varepsilon > 0$, let $U_\varepsilon : S^{N-1} \to S^{N-1}$ be such that

$$U_\varepsilon \equiv Q \quad \text{if } |x - P| \geq \frac{\varepsilon}{2} \quad \text{and} \quad \int_{S^{N-1}} |\nabla_T U_\varepsilon|^{N-1} \leq \sigma_N (N-1)^{\frac{N-1}{2}} + \varepsilon.$$

The existence of such map is established in [BCL, Section VIII].

Let W denote the ε-conic neighborhood of the geodesic segment joining P and N. We decompose W as $W = W_1 \cup W_2$, where W_1 is the cylindric part of W, and W_2 is the union of the two conic caps. We then define u_ε as

$$u_\varepsilon(x) = u(x) \quad \text{if } x \notin W, \quad u_\varepsilon(x) = U_\varepsilon(x) \quad \text{if } x \in W_1$$

and u_ε is extended by homogeneity of degree 0 in W_2. We then have

$$\int_{S^N} |D(u) - D(u_\varepsilon)| = \int_{W_1} |D(U_\varepsilon)| + o(1).$$

In [BCL, Section VIII], it is proved that

$$|D(U_\varepsilon)| = |\nabla_T U_\varepsilon| \leq \frac{1}{(N-1)^{\frac{N-1}{2}}} |\nabla_T U_\varepsilon|^{N-1}.$$

We conclude that

$$\lim_{\varepsilon \to 0} \int_{S^N} |D(u) - D(u_\varepsilon)| \leq \sigma_N L(u). \tag{D2}$$

On the other hand, since u_ε has only singularities of degree 0, Hopf's theorem implies that $u_\varepsilon \in \overline{C^\infty(S^N; S^{N-1})}^{W^{1,N-1}}$. It then follows from (D2) that inequality "$\leq$" in (D1) holds, at least when u is a good map.

We now establish $\leq$ in (D1) for any map $u \in W^{1,N-1}(S^N; S^{N-1})$. Let (u_n) be a sequence of good maps such that $u_n \to u$ in $W^{1,N-1}$. For each $n \geq 1$, we have just shown that there exists $v_n : S^N \to S^{N-1}$ smooth such that

$$\int_{S^N} |D(u_n) - D(v_n)| \leq \sigma_N L(u_n) + \frac{1}{n}.$$

Thus

$$\int_{S^N} |D(u) - D(v_n)| \leq \int_{S^N} |D(u) - D(u_n)| + \sigma_N L(u_n) + \frac{1}{n} = \sigma_N L(u) + o(1),$$

which gives the desired result.

Proof of Theorem 1″. We want to show that

$$E_{\mathrm{rel}}(u) = \int_{S^N} |\nabla u|^{N-1} + (N-1)^{\frac{N-1}{2}} \sigma_N L(u). \tag{D3}$$

Proof of "$\leq$". It suffices to establish the result for good maps. In fact, if $u_n \to u$ in $W^{1,N-1}$, then

$$E_{\mathrm{rel}}(u) \leq \liminf_{n \to \infty} E_{\mathrm{rel}}(u_n),$$

while the right-hand side of (D3) is continuous with respect to the strong topology in $W^{1,N-1}$. Thus, we may assume that u is good and we can proceed exactly as in the proof of Theorem 2″. We shall leave the details to the reader.

Proof of "$\geq$". As in [BCL, Section VIII], we have

$$\Big|(w \cdot v_2 \wedge \cdots \wedge v_N, w \cdot v_1 \wedge v_3 \wedge \cdots \wedge v_N, \dots)\Big| \leq \frac{|w| \left(\sum_j |v_j|^2 \right)^{\frac{N-1}{2}}}{(N-1)^{\frac{N-1}{2}}} \tag{D4}$$

for every $w, v_1, \dots, v_N \in \mathbb{R}^N$.

On the other hand, given a sequence $(u_n) \subset C^\infty(S^N; S^{N-1})$ such that (u_n) is bounded in $W^{1,N-1}$ and $u_n \to u$ a.e., we have

$$\text{(D5)} \qquad \int_{S^N} |\nabla u_n|^{N-1} = \int_{S^N} |\nabla u|^{N-1} + \int_{S^N} |\nabla u_n - \nabla u|^{N-1} + o(1).$$

Let $\zeta \in \operatorname{Lip}(S^N)$ be such that $\|\nabla \zeta\|_{L^\infty} \le 1$. Applying (D4) to $w = u_n$ and $v = \nabla u_n - \nabla u$, we get

$$\frac{\displaystyle\int_{S^N} |\nabla u_n - \nabla u|^{N-1}}{(N-1)^{\frac{N-1}{2}}} \ge \sum_\sigma \varepsilon_\sigma \int_{S^N} u_n \cdot (u_{nx_{\sigma 2}} - u_{x_{\sigma 2}}) \wedge \cdots \wedge (u_{nx_{\sigma N}} - u_{x_{\sigma N}})\, \zeta_{x_{\sigma 1}},$$

where the sum is taken with respect to all permutations σ of N elements, and $\varepsilon_\sigma = \pm 1$ denotes the sign of σ. Thus,

$$\text{(D6)} \qquad \int_{S^N} |\nabla u_n - \nabla u|^{N-1} \ge -(N-1)^{\frac{N-1}{2}} \int_{S^N} D(u) \cdot \nabla \zeta + R_n(\zeta).$$

Assume for the moment that $R_n(\zeta) \to 0$. It follows from (D5) and (D6) that

$$\liminf_{n\to\infty} \int_{S^N} |\nabla u_n|^{N-1} \ge \int_{S^N} |\nabla u|^{N-1} + (N-1)^{\frac{N-1}{2}} \langle T(u), \zeta \rangle.$$

Taking the supremum over ζ, we obtain "$\le$" in (D3).

Thus, in order to conclude the proof of Theorem $1''$, we need to show that $R_n(\zeta) \to 0$. Since each u_n is smooth,

$$\int_{S^N} D(u_n) \cdot \nabla \zeta = 0.$$

It then follows that

$$R_n(\zeta) = S_n(\zeta) + o(1),$$

where $S_n(\zeta)$ is a sum of integrals of the form

$$I_n = \pm \int_{S^N} u_n \cdot v_{nx_{i_1}} \wedge \cdots \wedge v_{nx_{i_k}} \wedge u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \zeta_{x_t},$$

where $k + l = N - 1$ and $v_n = u_n - u$. It is important to notice that $k \ge 1$ and $l \ge 1$. Since

$$(u_n - u)\, u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \to 0 \quad \text{strongly in } L^{\frac{N-1}{l}},$$

we may replace I_n by

$$\pm \int_{S^N} u \cdot v_{nx_{i_1}} \wedge \cdots \wedge v_{nx_{i_k}} \wedge u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \zeta_{x_t}.$$

We can now formally integrate by parts to write $S_n(\zeta)$ as a sum of integrals of the form

$$\pm \int_{S^N} u_{x_{i_1}} \cdot v_n \wedge v_{nx_{i_2}} \wedge \cdots \wedge v_{nx_{i_k}} \wedge u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \zeta_{x_t}.$$

Such integration by parts can be easily justified by approximation ; note that all the second derivatives are cancelled by symmetry.

Since

$$v_n\, u_{x_{j_1}} \wedge \cdots \wedge u_{x_{j_l}} \to 0 \quad \text{strongly in } L^{\frac{N-1}{l+1}},$$

we conclude that

$$R_n(\zeta) = S_n(\zeta) + o(1) = o(1).$$

The proof of Theorem $1'$ is completed.

E. Proof of Theorem $5''$.

Throughout this section, we assume that $\Omega = \partial G$, where G is a domain in $\mathbb{R}^N$; more generally, Ω could be any smooth domain (with boundary) in ∂G. We start with

Lemma E1. *Let $0 < \sigma < \infty$, $1 \le q < \infty$. Then*

$$\overline{C^\infty(\Omega; S^{N-1}) \cap W^{\sigma,q}}^{W^{\sigma,q}} = \overline{C^\infty(\overline{\Omega}; S^{N-1})}^{W^{\sigma,q}}.$$

In other words, for each map $u \in C^\infty(\Omega; S^{N-1}) \cap W^{\sigma,q}$, there is a sequence $(u_n) \subset C^\infty(\overline{\Omega}; S^{N-1})$ such that $u_n \to u$ in $W^{\sigma,q}$.

Proof. Let, for $t > 0$ sufficiently small, $\Omega_t = \big\{x \in \Omega\ ;\ d(x, \partial\Omega) > t\big\}$. Consider, for any such t, a diffeomorphism $\Phi_t : \Omega \to \Omega_t$ such that

$$\|D^k \Phi - D^k \mathrm{Id}\|_{L^\infty} \le C_k t, \quad k = 0, 1, 2 \ldots$$

and set, for $u \in C^\infty(\Omega; S^{N-1})$, $u^t = u \circ \Phi_t$. Then

$$u^t \in C^\infty(\overline{\Omega}; S^{N-1}) \qquad \text{and} \qquad u^t \to u \quad \text{in } W^{\sigma,q}.$$

Lemma E2. *Let $u \in W^{1,N}(\Omega; S^{N-1})$. Then*

$$L(u) = 0 \quad \Longleftrightarrow \quad u \in \overline{C^\infty(\Omega; S^{N-1}) \cap W^{1,N}}^{W^{1,N}}.$$

Here,

$$\langle T(u), \zeta\rangle = -\int_\Omega D(u) \cdot \nabla\zeta,$$

where $\zeta \in W^{1,\infty}(\Omega)$ and ζ is constant on each connected component of $\partial\Omega$; $L(u)$ is computed accordingly. When $N = 3$, this result is due to Bethuel [B1] ; the same proof yields Lemma E2.

Lemma E3. *Let $1 \le p < N$. For $g \in W^{1,p}(S^{N-1}; S^{N-1})$, set $\tilde{g}(x) = g\left(\frac{x}{|x|}\right)$ for $x \in B^N$. Then $\tilde{g} \in W^{1,p}$ and the map*

$$g \in W^{1,p}(S^{N-1}; S^{N-1}) \quad \longmapsto \quad \tilde{g} \in W^{1,p}(B^N; S^{N-1})$$

is continuous and verifies $|\tilde{g}|_{W^{1,p}} \le C|g|_{W^{1,p}}$.

Proof. Trivial computation.

Lemma E4. *In the definition of TV, we may replace $C^\infty(\overline{\Omega}; \mathbb{R}^N)$-maps by maps in* Lip $(\overline{\Omega}; \mathbb{R}^N)$.

Proof. Clear, by approximation.

Lemma E5. *Let $N \ge 2$, $N - 1 \le p < N$, and let $u \in W^{1,p}(\Omega; S^{N-1})$. Fix $a_1, \dots, a_k \in \Omega$ and define, for $\rho > 0$ sufficiently small,*

$$u_\rho(x) = \begin{cases} u(x), & \text{if } d(x, \{a_1, \dots, a_k\}) \ge \rho \\ u(y), & \text{if, for some } j,\ d(x, a_j) < \rho \text{ and} \\ & x \text{ lies on the geodesic segment from} \\ & a_j \text{ to } y, \text{ where } d(a_j, y) = \rho \end{cases}$$

(here, d denotes the geodesic distance in S^N.)
Then $u_{\rho_n} \to u$ for some sequence $\rho_n \to 0$.

Proof. For simplicity, we may assume Ω is flat near each a_j. Then the definition of u_ρ becomes

$$u_\rho(x) = \begin{cases} u(x) & \text{if } d(x, \{a_1, \dots, a_k\}) \ge \rho, \\ u\left(\rho \dfrac{x - a_j}{|x - a_j|}\right) & \text{if } |x - a_j| < \rho. \end{cases}$$

Alternatively, denoting by $u_j^\rho(y) = u(a_j + \rho y)$, $y \in S^{N-1}$, then

$$u_\rho(x) = u_j^\rho\left(\frac{x - a_j}{|x - a_j|}\right) \quad \text{if } |x - a_j| < \rho.$$

Assume, for simplicity, that there is only one singularity, say $a_1 = 0$.

Let, for $n \geq 1$, ρ_n be such that $\dfrac{1}{2n} \leq \rho_n \leq \dfrac{1}{n}$ and

$$\frac{1}{2n}\int_{S_{\rho_n}} |\nabla u|^p \leq \int_{\frac{1}{2n}}^{\frac{1}{n}} \left(\int_{|x|=\rho} |\nabla u|^p\right) d\rho = \int_{\frac{1}{2n} \leq |x| \leq \frac{1}{n}} |\nabla u|^p.$$

Then

$$\int_{B_{\rho_n}} |\nabla u_{\rho_n}|^p \leq C_{N,p}\rho_n \int_{S_{\rho_n}} |\nabla u|^p \leq C \int_{\frac{1}{2n} \leq |x| \leq \frac{1}{n}} |\nabla u|^p \overset{n\to\infty}{\longrightarrow} 0.$$

Thus

$$\begin{aligned}\int_\Omega |\nabla u_{\rho_n} - \nabla u|^p &= \int_{B_{\rho_n}} |\nabla u_{\rho_n} - \nabla u|^p \\ &\leq C\Big(\int_{B_{\rho_n}} |\nabla u_{\rho_n}|^p + \int_{B_{\rho_n}} |\nabla u|^p\Big) \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Proof of Theorem 5″. The equivalence

$$TV(u) < \infty \quad \Longleftrightarrow \quad \operatorname{Det}(\nabla u) \quad \text{is a measure}$$

is established as in the proof of Theorem 5. As already noted, $TV(u) < \infty$ implies

$$\operatorname{Det}(\nabla u) = \frac{\sigma_N}{N} \sum_{\text{finite}} (\delta_{P_i} - \delta_{N_i}).$$

Let $a_1, \ldots, a_k$ be the collection of points P_i, N_i. Given $n \geq 1$, let

$$\Omega^n = \left\{x \in \Omega \ ; \ d(x, \{a_1, \ldots, a_k\}) > \frac{1}{n}\right\}$$

and

$$A^n = \left\{x \in \Omega \ ; \ \frac{1}{n} \leq d(x, \{a_1, \ldots, a_k\}) \leq \frac{2}{n}\right\}.$$

Consider a sequence $(u_n^m) \subset C^\infty(\overline{\Omega^n}; S^{N-1})$ such that $u_n^m \to u$ in $W^{1,N-1}$ as $m \to \infty$ (clearly, $L(u|_{\Omega_n}) = 0$). There is some $\rho_n \in \left(\frac{1}{n}, \frac{2}{n}\right)$ such that, up to a subsequence in m,

(i) $u_n^m|_{\Sigma_{\rho_n}} \to u|_{\Sigma_{\rho_n}}$ in $W^{1,N-1}(\Sigma_{\rho_n})$ as $m \to \infty$,

(ii) $\displaystyle\int_{\Sigma_{\rho_n}} |\nabla u|^{N-1} \leq Cn \int_{A_n} |\nabla u|^{N-1}$.

Here, $\Sigma_\rho = \big\{x\ ;\ d(x, \{a_1, \dots, a_k\}) = \rho\big\}$.

Extend u_n^m to Ω as in Lemma 5 (by homogeneity of degree 0) ; let $(u_n^m)_{\rho_n} = \tilde{u}_n^m$ be this extension. By (i) and Lemma E5, we have $\tilde{u}_n^m \to u_{\rho_n}$ in $W^{1,N-1}(\Omega)$ as $m \to \infty$. By (ii) and Lemma E3, we have $u_{\rho_n} \to u$ in $W^{1,N-1}(\Omega)$. Thus we may find a sequence $(v_n) \subset \mathrm{Lip}_{\mathrm{loc}}\,(\Omega \setminus \{a_1, \dots, a_k\}; S^{N-1})$ such that

(a) v_n is homogeneous of degree 0 near each a_j ;

(b) $v_n \to u$ in $W^{1,N-1}(\Omega)$;

(c) near each a_j, the degree of v_n is d_j.

Assertion (c) follows from (i), the continuity of degree of maps from S^{N-1} into S^{N-1} for $W^{1,N-1}$ convergence and the following clear fact

$$T(v_n) = \sigma_N \sum_j \deg\,(v_n, a_j)\delta_{a_j} \rightharpoonup T(u) = \sigma_N \sum_j d_j \delta_{a_j}.$$

For the remaining part of the proof, assume for simplicity that there is only one singularity $a = 0$ of degree $d > 0$ and that Ω is flat near a. Let ρ_n be such that v_n is homogenous of degree 0 in $B_{\rho_n}(0)$. Here, $B_\rho(0)$ is a ball in Ω centered at $a = 0$. Fix d distinct points $p_1, \dots, p_d$ in B_1.

Let $\varepsilon > 0$ be sufficiently small. For $w \in \mathrm{Lip}\,(\partial B_1; S^{N-1})$ (B_1 is the unit ball in $\mathbb{R}^N$), with $\deg w = d$, let $\tilde{w} : B_1 \setminus \bigcup_j B_\varepsilon(p_j) \to S^{N-1}$ be such that $\tilde{w}|_{\partial B_1} = w$, $\tilde{w}(x) = \dfrac{x - p_j}{|x - p_j|}$ if $|x - p_j| = \varepsilon$, $j = 1, \dots, d$ and $\tilde{w}$ is a Lipschitz function. (Such a map exists, since $\deg w = d$.) We then extend $\tilde{w}$ to B_1 by setting $\tilde{w}(x) = \frac{x - p_j}{\varepsilon}$, if $|x - p_j| \leq \varepsilon$. Thus $\tilde{w}$ is still Lipschitz.

Define, for $0 < \rho < \rho_n$,

$$v_{n,\rho}(x) = \begin{cases} v_n(x), & \text{if } d(x,0) > \rho \\ \tilde{w}_n\left(\dfrac{x}{\rho}\right), & \text{if } d(x,0) \leq \rho \end{cases};$$

here, $w_n(x) = v_n(\rho_n x)$, if $|x| = 1$. Lemma E5 yields

$$v_{n,\rho} \to v_n \quad \text{in } W^{1,N-1} \quad \text{as } \rho \to 0.$$

Clearly, by the definition of $v_{n,\rho}$, we have

$$\int_\Omega |\operatorname{Jac} v_{n,\rho}| = \omega_N d.$$

Considering now the case of several singularities, we obtain by a diagonal procedure a sequence $(w_n) \subset \operatorname{Lip}(\Omega; \mathbb{R}^N)$ such that

$$w_n \to u \quad \text{in } W^{1,N-1} \quad \text{and} \quad \int_\Omega |\operatorname{Jac} w_n| = \omega_N \sum_j |d_j|.$$

F. Proof of Theorems 5‴.

We start with the following well-known

Lemma F1. *Let $1/p < s < 1$ and $1 < p < \infty$. Given $u \in W_0^{1,p}(\mathbb{R}^N_+)$, let*

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x_N > 0, \\ 0 & \text{if } x_N \leq 0. \end{cases}$$

Then, $u \mapsto \tilde{u}$ is a continuous mapping from $W_0^{1,p}(\mathbb{R}^N_+)$ into $W^{1,p}(\mathbb{R}^N)$.

Proof. By density, it suffices to deal with $u \in C_0^\infty(\mathbb{R}^N_+)$. Using the Besov seminorm, we have

$$\text{(F1)} \quad |\tilde{u}|^p_{W^{s,p}} \sim |u|^p_{W^{s,p}} + \|u\|^p_{L^p} + \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \frac{|\tilde{u}(x', t+\tau) - \tilde{u}(x', t)|^p}{\tau^{1+sp}}\, dx'\, dt\, d\tau.$$

Denote by I the last term in the right-hand side of (F1). Clearly, it suffices to estimate I :

(F2)

$$I \sim \int_{\mathbb{R}^{N-1}} \int_0^\infty \int_0^\infty \frac{|u(x', t) - u(x', \sigma)|^p}{|t - \sigma|^{1+sp}}\, d\sigma\, dt\, dx' + \int_{\mathbb{R}^{N-1}} \int_0^\infty \frac{|u(x', t)|^p}{t^{sp}}\, dt\, dx'.$$

It then suffices to estimate the last term in (F2). By Fubini, it suffices to consider the 1-dimensional integral

$$\int_0^\infty \frac{|u(t)|^p}{t^{sp}}\, dt, \quad u \in C_0^\infty(0, \infty).$$

This is a trivial consequence of the following

Lemma F2. *Let $1/p < s < 1$ and $1 < p < \infty$. Given $u \in C_0^\infty(0,\infty)$, we have*

$$\int_0^\infty \frac{|u(t)|^p}{t^{sp}}\,dt \le C \int_0^\infty\!\!\int_0^\infty \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}}\,dx\,dy. \tag{F3}$$

Proof. We first point out that both integrals are finite. Given $0<\alpha<\beta<1$, we have

$$|u(t)| \le |u(x)| + |u(t)-u(x)| \quad \forall x \in [\alpha t, \beta t] =: I_t.$$

Thus,

$$|u(t)|^p \le 2|u(x)|^p + C|u(t)-u(x)|^p \quad \forall x \in I_t.$$

Integrating over $x \in I_t$, we get

$$|u(t)|^p \le \frac{2}{(\beta-\alpha)t}\int_0^\infty |u(x)|^p\,dx + \frac{C}{(\beta-\alpha)t}\int_{I_t} |u(t)-u(x)|^p\,dx. \tag{F4}$$

Since $\alpha,\beta<1$, we have $|t-x| \sim t$ for every $x \in I_t$. Integrating (F4) with respect to t we then have

$$\begin{aligned}\int_0^\infty \frac{|u(t)|^p}{t^{sp}}\,dt &\le \frac{2}{\beta-\alpha}\int_0^\infty\!\!\int_{I_t} \frac{|u(x)|^p}{t^{sp}}\,dx\,dt + \frac{C}{\beta-\alpha}\int_0^\infty\!\!\int_0^\infty \frac{|u(t)-u(x)|^p}{|t-x|^{1+sp}}\,dx\,dt\\ &= \frac{2(\beta^{sp}-\alpha^{sp})}{sp(\beta-\alpha)}\int_0^\infty \frac{|u(x)|^p}{x^{sp}}\,dx + \frac{C}{\beta-\alpha}\int_0^\infty\!\!\int_0^\infty \frac{|u(t)-u(x)|^p}{|t-x|^{1+sp}}\,dx\,dt. \qquad \text{(F5)}\end{aligned}$$

Note that $\frac{\beta^{sp}-\alpha^{sp}}{sp(\beta-\alpha)} = \gamma^{sp-1}$ for some $\gamma \in [\alpha,\beta]$. Thus, by taking $\beta \ll 1/2$ sufficiently small, we have

$$\frac{2(\beta^{sp}-\alpha^{sp})}{sp(\beta-\alpha)} \le \frac{1}{2}.$$

With such choice, (F3) trivially follows from (F5).

Lemma F3. *Let ω_1, Ω be two smooth domains, $\omega_1 \subset\subset \Omega$. Set $\omega_2 = \Omega\backslash\overline{\omega}_1$. Assume that $u_n \to u$ in $W^{s,p}(\omega_1)$ and $v_n \to v$ in $W^{s,p}(\omega_2)$, with $\operatorname{tr} u_n = \operatorname{tr} v_n$ on $\partial\omega_1$. Let*

$$w_n = \begin{cases} u_n & \text{in } \omega_1\\ v_n & \text{in } \omega_2\end{cases} \quad \text{and} \quad w = \begin{cases} u & \text{in } \omega_1\\ v & \text{in } \omega_2\end{cases}.$$

Then

$$w_n \to w \quad \text{in } W^{s,p}(\Omega).$$

Proof. It suffices to show that

$$\|w\|_{W^{s,p}(\Omega)} \le C\big(\|u\|_{W^{s,p}(\omega_1)} + \|v\|_{W^{s,p}(\omega_2)}\big).$$

Let $\eta = \operatorname{tr} u = \operatorname{tr} v$. By the standard trace theory,

(i) $\|\eta\|_{W^{s-1/p,p}} \leq C\|u\|_{W^{s,p}(\omega_1)}$;

(ii) $\|\eta\|_{W^{s-1/p,p}} \leq C\|v\|_{W^{s,p}(\omega_2)}$;

(iii) there exists an extension $g \in W^{s,p}(\Omega)$ of η to ω_1 and ω_2 such that

$$\|g\|_{W^{s,p}(\Omega)} \leq C\|\eta\|_{W^{s-1/p,p}}.$$

Let

$$\tilde{w} = \begin{cases} u - g & \text{in } \omega_1, \\ v - g & \text{in } \omega_2. \end{cases}$$

By Lemma F1, $\tilde{w} \in W^{s,p}(\Omega)$ and

$$\|\tilde{w}\|_{W^{s,p}(\Omega)} \leq C\big(\|u\|_{W^{s,p}(\omega_1)} + \|v\|_{W^{s,p}(\omega_2)}\big).$$

Since $w = \tilde{w} + g$ a.e. in Ω, Lemma F.3 follows.

In the sequel, we shall denote by C the cube $(-1,1)^N$. Let $\|x\|_\infty = \operatorname{Max}_i \{|x_i|\}$.

Lemma F4 (Brezis-Mironescu [BM1]). *Let $0 < s < 1$ and $1 < p < \infty$, with $sp < N$. Given $f \in W^{s,p}(\partial C)$, set $\tilde{f}(x) = f\big(x/\|x\|_\infty\big)$, $x \in C$. Then, $\tilde{f} \in W^{s,p}(C)$ and the mapping*

$$f \mapsto \tilde{f}$$

is continuous from $W^{s,p}(\partial C)$ into $W^{s,p}(C)$.

We refer the reader to [BM1, Lemma D.1] for a proof of Lemma F4.

We denote by $C_\varepsilon = (-\varepsilon, \varepsilon)^N$. The following lemma is a variant of the approximation procedure in [BM2]:

Lemma F5. *Let $0 < s < 1$ and $1 < p < \infty$, with $sp < N$. Let $f \in W^{s,p}(\mathbb{R}^N)$. Given $\varepsilon > 0$ and $Q \in C_\varepsilon$, set*

$$f_{\varepsilon,Q}(x) = \begin{cases} f(x) & \text{if } x \notin C_\varepsilon(Q), \\ f(\pi_{\varepsilon,Q}(x)) & \text{if } x \in C_\varepsilon(Q), \end{cases}$$

where $\pi_{\varepsilon,Q}(x) = Q + \varepsilon\frac{x-Q}{\|x-Q\|_\infty}$ is the projection of x to $\partial C_\varepsilon(Q)$, with respect to Q. Then, there exist $\varepsilon_n \to 0$ and $Q_n \in C_{\varepsilon_n/2}$ such that

$$f_{\varepsilon_n,Q_n} \to f \quad \text{in } W^{s,p}(\mathbb{R}^N).$$

Proof. Set $g_{\varepsilon,Q} = f_{\varepsilon,Q} - f$. We have

$$\begin{aligned}
|g_{\varepsilon,Q}|^p_{W^{s,p}} &= 2\int_{\mathbb{R}^N\setminus C_\varepsilon(Q)} dy \int_{C_\varepsilon(Q)} \frac{|f_{\varepsilon,Q}(x)-f(x)|^p}{|x-y|^{N+sp}}\,dx + \\
&\qquad\qquad + \int_{C_\varepsilon(Q)}\int_{C_\varepsilon(Q)} \frac{|g_{\varepsilon,Q}(x)-g_{\varepsilon,Q}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy \\
&\sim \int_{C_\varepsilon(Q)} \frac{|f_{\varepsilon,Q}(x)-f(x)|^p}{d(x,\partial C_\varepsilon(Q))^{sp}}\,dx + \\
&\qquad\qquad + \int_{C_\varepsilon(Q)}\int_{C_\varepsilon(Q)} \frac{|f_{\varepsilon,Q}(x)-f_{\varepsilon,Q}(y)|^p}{|x-y|^{N+sp}}\,dx\,dy + o(1) \\
&=: I_{\varepsilon,Q} + J_{\varepsilon,Q} + o(1).
\end{aligned}$$

It suffices to show that

$$\int_0^1 \frac{d\varepsilon}{\varepsilon}\left\{ \fint_{C_\varepsilon} (I_{\varepsilon,Q}+J_{\varepsilon,Q})\,dQ \right\} < \infty.$$

The proof of

$$\int_0^1 \frac{d\varepsilon}{\varepsilon}\fint_{C_\varepsilon} J_{\varepsilon,Q}\,dQ < \infty$$

may be found in [BM2, Appendix A]. Next,

$$\int_0^1 \frac{d\varepsilon}{\varepsilon}\fint_{C_\varepsilon} I_{\varepsilon,Q}\,dQ \le \int_0^1 \frac{d\varepsilon}{\varepsilon}\int_{C_{2\varepsilon}} dx \fint_{C_\varepsilon(x)} \frac{|f_{\varepsilon,Q}(x)-f(x)|^p}{d(x,\partial C_\varepsilon(Q))^{sp}}\,dQ.$$

We now make the change of variables $Q = x - y$, $y \in C_\varepsilon$. We get

$$\int_0^1 \frac{d\varepsilon}{\varepsilon}\fint_{C_\varepsilon} I_{\varepsilon,Q}\,dQ \le \int_0^1 \frac{d\varepsilon}{\varepsilon}\int_{C_{2\varepsilon}} dx \fint_{C_\varepsilon} \frac{|f(x)-f(x-y+\varepsilon\frac{y}{\|y\|})|^p}{|y-\varepsilon\frac{y}{\|y\|}|^{sp}}\,dy.$$

Let $z = -y + \varepsilon\frac{y}{\|y\|}$, whose Jacobian is $O((\frac{\varepsilon}{\|z\|^{N-1}}))$ (see [BM2]). We then have

$$\begin{aligned}
\int_0^1 \frac{d\varepsilon}{\varepsilon}\fint_{C_\varepsilon} I_{\varepsilon,Q}\,dQ &\le C\int_0^1 \frac{d\varepsilon}{\varepsilon}\int_{C_{2\varepsilon}} dx \fint_{C_\varepsilon} \frac{|f(x)-f(x+z)|^p}{|z|^{N+sp-1}}\varepsilon^{N-1}\,dz \\
&\le C\int_{C_2} dx \int_{C_1} \frac{|f(x)-f(x+z)|^p}{|z|^{N+sp-1}}\,dz \int_{|z|}^1 \frac{d\varepsilon}{\varepsilon^2} \\
&\le \iint \frac{|f(x)-f(x+z)|^p}{|z|^{N+sp}}\,dx\,dz < \infty.
\end{aligned}$$

This concludes the proof of Lemma F5.

Given $N-1 < p \le N$, any map $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$ has a harmonic extension $U \in W^{N/p,p}(B^{N+1}; \mathbb{R}^N) \subset W^{1,N}$. We then define $T(u)$ as

$$\langle T(u), \zeta \rangle = -\sum_{j=1}^{N} \int_{B^{N+1}} \det\left(U_{x_1}, \ldots, U_{x_j}, u, U_{x_{j+1}}, \ldots, U_{x_{N+1}}\right) \xi_{x_j},$$

where $\zeta \in \mathrm{Lip}\,(S^N)$ and ξ is any extension of ζ to B^{N+1}. One can see that this definition is independent of the extension ξ. Let

$$L(u) = \frac{1}{\sigma_N} \operatorname*{Max}_{\|\nabla\zeta\|_{L^\infty} \le 1} \langle T(u), \zeta \rangle.$$

We have the following

Lemma F6. *Assume $N \ge 2$. Let $1 < p < \infty$ and $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$. If $T(u) = 0$, then $u \in \overline{C^\infty(S^N; S^{N-1})}^{W^{(N-1)/p,p}}$.*

Proof.
Case 1. Proof of the lemma if $N \ge 3$.

Note that good maps are dense in $W^{1,N-1}(S^N; S^{N-1})$ and, by interpolation, in $W^{(N-1)/p,p}(S^N; S^{N-1})$. Thus, it suffices to show that if u is a good map, then there exists $v \in C^\infty(S^N; S^{N+1})$ such that

$$\|u - v\|_{W^{1,N-1}} \le C L(u),$$

which can be done by a dipole construction. By interpolation, we obtain Lemma F6.
Case 2. Proof of the lemma if $N = 2$.

The interpolation argument does not work in this case. However, for any map $u \in \mathcal{R}$, the dipole construction in [BBM2] gives a sequence (v_n) such that

$$T(v_n) = T(u), \quad |v_n|^p_{W^{1/p,p}} \le C L(u), \quad \text{and} \quad v_n \to 1 \quad \text{a.e.}$$

Clearly, $u\bar{v}_n \in \overline{C^\infty}^{W^{1/p,p}}$ and

$$\begin{aligned} |u - u\bar{v}_n|^p_{W^{1/p,p}} &\le 2|1 - \bar{v}_n|^p_{W^{1/p,p}} + K \iint |1 - \bar{v}_n(x)|^p \frac{|u(x) - u(y)|^p}{|x-y|^2}\, dx\, dy \\ &\le 2CL(u) + o(1). \end{aligned}$$

Using the density of $\mathcal{R}$ in $W^{1/p,p}(S^2; S^1)$, we obtain the desired result.

We also point out the following extension of Lemma F6 whose proof is left to the reader :

Lemma F7. *Assume $N-1 < p \le N$. Let Ω be a smooth subdomain of S^N. For any $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$, $T(u|_\Omega)$ is well-defined when computed against Lipschitz functions which are constant on each connected component of $\partial\Omega$. If $T(u|_\Omega) = 0$, then $u|_\Omega \in \overline{C^\infty(\overline{\Omega}; S^{N-1})}^{W^{(N-1)/p,p}}$.*

A key ingredient in the proof of Theorem 5$'''$ is the following

Proposition F1. *Assume $N-1 < p \le N$. Let $u \in W^{(N-1)/p,p}(S^N; S^{N-1})$ be such that*

$$T(u) = \sigma_N \sum_{\text{finite}} d_i \delta_{M_i}.$$

Then, there exist $(u_n) \subset W^{(N-1)/p,p}(S^N; S^{N-1})$, $\varepsilon_n \to 0$, and $M_i^n \to M_i$ such that

(i) $u_n \in C^\infty(S^N \backslash \bigcup_i C_{\varepsilon_n}(M_i^n); S^{N-1})$;

(ii) u_n *is homogeneous of degree* 0 *on each cube* $C_{\varepsilon_n}(M_i^n)$;

(iii) $T(u_n) = \sigma_N \sum_{\text{finite}} d_i \delta_{M_i^n}$;

(iv) $u_n \to u$ *in* $W^{(N-1)/p,p}$.

Proof. We first observe that the set of good pairs (ε, Q), in the sense of Lemma F5, is "fat". More precisely, there exists a sequence $\varepsilon_n \to 0$ such that

$$\frac{\left|\left\{Q \in C_{\varepsilon_n/4} \; ; \; |f_{\varepsilon_n,Q} - f|_{W^{(N-1)/p,p}} < 1/n\right\}\right|}{|C_{\varepsilon_n/4}|} \ge \frac{1}{2}.$$

In particular, $\left\{Q \in C_{\varepsilon_n/4} \; ; \; |f_{\varepsilon_n,Q} - f|_{W^{(N-1)/p,p}} < 1/n\right\}$ intersects the complement of any null set of $C_{\varepsilon_n/4}$. For $n \ge 1$ fixed, consider

$$v_n = u|_{S^N \backslash \bigcup_i C_{\varepsilon_n/4}(M_i)}.$$

Then, $T(v_n) = 0$, so that there exists a sequence $(v_n^k)_k$,

$$v_n^k \in C^\infty\big(S^N \backslash \bigcup_i C_{\varepsilon_n/4}(M_i); S^{N-1}\big) \quad \forall k \ge 1,$$

such that

$$v_n^k \to v_n \quad \text{in } W^{(N-1)/p,p}\big(S^N \backslash \bigcup_i C_{\varepsilon_n/4}(M_i)\big) \quad \text{as } k \to \infty.$$

By Fubini, for a.e. $Q \in C_{\varepsilon_n/4}$, we have

$$v_n^k|_{\bigcup_i \partial C_{\varepsilon_n}(Q+M_i)} \to u|_{\bigcup_i \partial C_{\varepsilon_n}(Q+M_i)} \quad \text{in } W^{(N-1)/p,p} \quad \text{as } k \to \infty.$$

By Lemmas F3 and F4, for any such Q we have

$$\tilde{v}_n^k \to \tilde{u}_n \quad \text{in } W^{(N-1)/p,p} \quad \text{as } k \to \infty,$$

where $\tilde{v}_n^k$ (resp., $\tilde{u}_n$) is v_n^k (resp., u) extended by homogeneity of degree 0 on each cube $C_{\varepsilon_n}(Q + M_i)$. By Lemma F5, we can choose $Q = Q_n$ such that $\tilde{u}_n \to u$ in $W^{(N-1)/p,p}$. Applying a diagonalization argument, u_n may be taken among $(\tilde{u}_n^k)$. We only need to show that (iii) holds.

Note that $\tilde{v}_n^k$ is locally Lipschitz on $S^N \setminus \bigcup_i M_i^n$, where $M_i^n = Q_n + M_i$. Thus,

$$T(\tilde{v}_n^k) = \sigma_N \sum_{\text{finite}} \tilde{d}_i \delta_{M_i^n}.$$

Since $T(u_n) \rightharpoonup T(u)$, for n large enough we have (iii). This concludes the proof of Proposition F1.

We may now present the

Proof of Theorem 5'''.

Step 1. If $TV(u) < \infty$, then $\operatorname{Det}(\nabla u)$ is a measure and

$$|\operatorname{Det}(\nabla u)|_{\mathcal{M}} \leq TV(u). \tag{F6}$$

Clearly, we may replace u_n in the definition of $TV(u)$ by

$$\tilde{u}_n = \begin{cases} u_n & \text{if } |u_n| \leq 1, \\ \frac{u_n}{|u_n|} & \text{if } |u_n| > 1. \end{cases}$$

We may thus assume that $|u_n| \leq 1$. Since $u_n \to u$ in $W^{(N-1)/p,p}$ and $|u_n| \leq 1$, we have

$$\langle T(u_n), \zeta \rangle \to \langle T(u), \zeta \rangle$$

for every $\zeta \in \operatorname{Lip}(S^N; S^{N-1})$. In addition,

$$\langle T(u_n), \zeta \rangle = N \int_{S^{N-1}} \operatorname{Jac} u_n \, \zeta \leq TV(u) \|\zeta\|_{L^\infty} + o(1).$$

Thus, $T(u)$ is a measure and (F6) holds.

Step 2. If $\operatorname{Det}(\nabla u)$ is a measure, then $TV(u) < \infty$ and

$$\frac{1}{\omega_n} TV(u) = \text{number of topological singularities of } u. \tag{F7}$$

By Proposition F1, it suffices to compute $TV(u)$, where u is smooth outside finitely many (disjoint) cubes and u is homogeneous of degree 0 inside each one of these cubes. By (F6), we have "$\geq$" in (F7). It then suffices to show the reverse inequality.

Note that $u \in W^{1,q}(S^N; S^{N-1})$ for every $q < N$. As in the proof of the case $W^{1,N-1}(S^N; S^{N-1})$, we can find $(u_n) \subset C^\infty$, $u_n \to u$ in $W^{1,q}$, with

$$\frac{1}{\omega_N} \int_{S^{N-1}} |\operatorname{Jac} u_n| = \text{number of topological singularities of } u.$$

For $N - 1 < q < N$, we have $W^{1,q} \cap L^\infty \subset W^{(N-1)/p,p}$, so that $u_n \to u$ in $W^{(N-1)/p,p}$. We conclude that (F7) holds.

G. Proofs of Theorems 14–17.

We start by establishing the precise value of $E(g)$:

Lemma G1. *Let $g \in BV(I; S^1)$ and let A be the set of jump points of g. Then*

$$E(g) = |\dot{g}_d| + \sum_{a \in A} d_{S^1}(g(a+), g(a-)). \tag{G1}$$

Proof. Let $\varphi \in BV(I; \mathbb{R})$ be any lifting of g. We claim that

$$|\dot{\varphi}_d|_{\mathcal{M}(I)} = |\dot{g}_d|_{\mathcal{M}(I)}. \tag{G2}$$

Indeed, recall that, by the chain rule, we have

$$\dot{\varphi}_d = -i\bar{g}\dot{g}_d.$$

Set $\nu = \dot{g}_d$ and $\mu = |\nu|$. Then there is some $k \in L^\infty((I, d\mu); S^1)$ such that $\nu = k\mu$. Since ν is diffuse and $-i\bar{g} \in BV$, we have $-ig \in L^\infty((I, d\mu); S^1)$, and thus $\dot{\varphi}_d = \ell\mu$, where $\ell = -i\bar{g}k \in L^\infty((I, d\mu); S^1)$. It follows that

$$|\dot{\varphi}_d|_{\mathcal{M}(I)} = \operatorname*{Sup}_{\substack{\zeta \in C_0(I;\mathbb{C}) \\ |\zeta| \leq 1}} \langle \dot{\varphi}_d, \zeta \rangle = \operatorname*{Sup}_{\substack{\zeta \in C_0(I;\mathbb{C}) \\ |\zeta| \leq 1}} \langle \mu, \ell\zeta \rangle = \langle \mu, |\ell| \rangle = |\mu|_{\mathcal{M}(I)} = |\dot{g}_d|_{\mathcal{M}(I)}.$$

Let now B denote the set of the jump points of φ; clearly, $B \supset A$. For each $a \in A$, we have

$$\begin{cases} g(a+) = \mathrm{e}^{i\varphi(a+)}, \\ g(a-) = \mathrm{e}^{i\varphi(a-)}, \end{cases}$$

so that

$$|\varphi(a+) - \varphi(a-)| \geq d_{S^1}(g(a+), g(a-)).$$

If $a \in B \setminus A$, then $\mathrm{e}^{i\varphi(a+)} = \mathrm{e}^{i\varphi(a-)}$; thus

$$|\varphi(a+) - \varphi(a-)| \geq 2\pi.$$

Consequently,

$$\tag{G3} \begin{aligned} |\dot{\varphi}|_{\mathcal{M}(I)} &= |\dot{\varphi}_d|_{\mathcal{M}(I)} + \sum_{a \in A} |\varphi(a+) - \varphi(a-)| + \sum_{a \in B \setminus A} |\varphi(a+) - \varphi(a-)| \\ &\geq |\dot{g}_d| + \sum_{a \in A} d_{S^1}(g(a+), g(a-)) + 2\pi \operatorname{card}(B \setminus A) \\ &\geq |\dot{g}_d| + \sum_{a \in A} d_{S^1}(g(a+), g(a-)). \end{aligned}$$

Equality holds if and only if

$$|\varphi(a+) - \varphi(a-)| = d_{S^1}(g(a+), g(a-)) \quad \forall a \in A \quad \text{and} \quad B = A.$$

This proves "$\geq$" in (G1).

In order to prove "$\leq$", we split A as $A = A_1 \cup A_2$, where

$$A_1 = \big\{a \in A; |g(a+) - g(a-)| = 2\big\} \quad \text{and} \quad A_2 = \big\{a \in A; |g(a+) - g(a-)| < 2\big\}.$$

If $a \in A_2$ we may define a *signed* distance

$$\delta_{S^1}(g(a+), g(a-)) = \arg\left(\frac{g(a+)}{g(a-)}\right).$$

Here, arg stands for the argument in $(-\pi, \pi)$. Set

$$\tag{G4} \mu = -i\bar{g}\dot{g}_d + \pi \sum_{a \in A_1} \delta_a + \sum_{a \in A_2} \delta_{S^1}(g(a+), g(a-))\, \delta_a.$$

We claim that μ is a measure. Indeed, A_1 is finite, since $g \in BV$. On the other hand,

$$\operatorname*{Sup}_{a \in A_2} |g(a+) - g(a-)| = d < 2$$

(again since $g \in BV$). Thus there is some $C > 0$ such that

$$\left|\delta_{S^1}(g(a+), g(a-))\right| \le C|g(a+) - g(a-)|.$$

It follows that

$$\sum_{a \in A_2} \left|\delta_{S^1}(g(a+), g(a-))\right| \le C \sum_{a \in A_2} |g(a+) - g(a-)| < \infty.$$

Assume that $I = (0, \alpha)$ for some $\alpha > 0$ and set $\varphi_0(x) = \mu((0, x))$, $x \in I$. We claim that, up to a constant, φ_0 is a lifting of g and that $|\dot{\varphi}_0|_{\mathcal{M}(I)} = E(g)$.

Indeed, using the chain rule for a product we have

$$\text{(G5)} \qquad \overline{g\mathrm{e}^{-i\varphi_0}}^{\,\cdot} = \mathrm{e}^{-i\varphi_0}\dot{g}_d - ig\,\mathrm{e}^{-i\varphi_0}(\dot{\varphi}_0)_d + \sum_{a \in A} \left(g\mathrm{e}^{-i\varphi}(a+) - g\mathrm{e}^{-i\varphi_0}(a-)\right)\delta_a.$$

Here, we have used the fact that φ is continuous outside A. For $a \in A$, we have

$$\varphi_0(a+) = \varphi_0(a-) + \mu(\{a\}),$$

so that

$$\mathrm{e}^{-i\varphi_0(a+)} = \mathrm{e}^{-i\varphi_0(a-)}\frac{g(a-)}{g(a+)},$$

by our definition of μ. Thus the sum in (G5) vanishes. On the other hand,

$$(\dot{\varphi}_0)_d = -i\bar{g}\dot{g}_d,$$

so that

$$\mathrm{e}^{-i\varphi_0}\dot{g}_d - ig\mathrm{e}^{-i\varphi_0}(\dot{\varphi}_0)_d = e^{-i\varphi_0}(\dot{g}_d - \dot{g}_d) = 0.$$

Thus, there is some $C \in \mathbb{C}$ such that $\varphi = \varphi_0 + C$ is a lifting of g.

On the other hand,

$$\begin{aligned} |\dot{\varphi}|_{\mathcal{M}(I)} = |\dot{\varphi}_0|_{\mathcal{M}(I)} &= |-i\bar{g}\dot{g}_d|_{\mathcal{M}(I)} + \pi\,\mathrm{card}\,(A_1) + \sum_{a \in A_2} \left|\delta_{S^1}(g(a+), g(a-))\right| \\ &= |\dot{g}_d|_{\mathcal{M}(I)} + \sum_{a \in A} d_{S^1}(g(a+), g(a-)). \end{aligned}$$

The proof of Lemma G1 is complete.

Proof of Theorem 14. We shall prove a slightly stronger assertion, which implies all the properties stated in the theorem. Namely, a lifting $\varphi \in BV(I; \mathbb{R})$ of g is a canonical lifting if and only if

(i) φ and g have the same jump sets ;

(ii) for $a \in A_1$, we have $\varphi(a+) - \varphi(a-) = \pm\pi$;

(iii) for $a \in A_2$, we have $\varphi(a+) - \varphi(a-) = \delta_{S^1}(g(a+), g(a-))$.

Property (i) was seen to be necessary for optimality in the proof of Lemma G1. Recall that, in addition to (i), equality in (G3) amounts to

$$|\varphi(a+) - \varphi(a-)| = d_{S^1}(g(a+), g(a-)) \quad \forall a \in A. \tag{G6}$$

If $a \in A_1$, then

$$|\varphi(a+) - \varphi(a-)| = \pi,$$

so that (ii) holds. Assume $a \in A_2$. Since $\mathrm{e}^{i\varphi(a+)} = g(a+)$ and $\mathrm{e}^{i\varphi(a-)} = g(a-)$, then by (G6) we have

$$\varphi(a+) - \varphi(a-) = \arg\left(\frac{g(a+)}{g(a-)}\right),$$

which gives (iii). Conversely, it is easy to see that, if (i)–(iii) are fullfilled, then equality holds in (G3).

Proof of Theorem 15. We identify $S^1 \setminus \{z\}$ with an interval I. Let A, A_1, A_2 be defined as in the proof of Lemma G1. We claim that, for *each* choice of integers $\varepsilon_a \in \{-1, 1\}$, $a \in A_1$, there is a canonical lifting φ of g on I such that

$$\varphi(a+) - \varphi(a-) = \varepsilon_a \pi \quad \forall a \in A_1.$$

This φ is obtained as in the proof of Lemma G1, as $\varphi = \mu((0, x)) + C$, $x \in I$. One simply has to modify the definition of μ by taking

$$\mu = -i\bar{g}\dot{g}_d + \sum_{a \in A_1} \varepsilon_a \pi \delta_a + \sum_{a \in A_2} \delta_{S^1}(g(a+), g(a-))\, \delta_a.$$

Moreover, the proof of Theorem 14 shows that, by this procedure, we obtain *all* canonical liftings. We claim that if φ is the canonical lifting corresponding to the choice ε_a, $a \in A_1$, and $\tilde{\varphi}$ the one corresponding to $\tilde{\varepsilon}_a$, $a \in A_1$, then

$$\frac{\tilde{\varphi}(z-) - \tilde{\varphi}(z+)}{2\pi} = \frac{\varphi(z-) - \varphi(z+)}{2\pi} + \frac{1}{2} \sum_{a \in A_1} (\tilde{\varepsilon}_a - \varepsilon_a) \in \mathbb{Z}.$$

If we identify $S^1 \setminus \{z\}$ with $I = (0, \alpha)$, $\alpha > 0$, this amounts to proving that

$$\frac{\tilde{\varphi}(\alpha-) - \tilde{\varphi}(0+)}{2\pi} = \frac{\varphi(\alpha-) - \varphi(0+)}{2\pi} + \frac{1}{2} \sum_{a \in A_1} (\tilde{\varepsilon}_a - \varepsilon_a).$$

We have $\tilde{\varphi}(\alpha-) - \tilde{\varphi}(0+) = \tilde{\mu}(I)$ (where $\tilde{\mu}$ is the corresponding measure); a similar assertion holds for φ. Thus

$$\frac{\tilde{\varphi}(\alpha-) - \tilde{\varphi}(0)}{2\pi} - \frac{\varphi(\alpha-) - \varphi(0+)}{2\pi} = \frac{(\mu - \tilde{\mu})(I)}{2\pi} = \frac{1}{2} \sum_{a \in A_1} (\tilde{\varepsilon}_a - \varepsilon_a).$$

Let φ be the canonical lifting corresponding to the choice $\varepsilon_a = -1$, $a \in A_1$. Then it is clear that, with $d = \frac{\varphi(z-)-\varphi(z+)}{2\pi}$ and $k = \operatorname{card} A_1$, we have

$$\operatorname{Deg}_1 g = \{d, d+d, \dots, d+k\}.$$

We next prove that d is an integer. This follows easily from the fact that $e^{i\varphi(z-)} = e^{i\varphi(z+)}$, so that $\varphi(z-) - \varphi(z+) \in 2\pi\mathbb{Z}$.

It remains to establish that $\operatorname{Deg}_1 g$ does not depend on the choice of z. Let w be any other continuity point of g. Let ψ be a canonical lifting of g on $S^1 \setminus \{w\}$. Since g is continuous at w, there is some $k \in \mathbb{Z}$ such that $\psi(w-) = \psi(w+) + 2k\pi$. We set

$$\varphi(\xi) = \begin{cases} \psi(\xi) & \text{if } \xi \in (w+, z-), \\ \psi(\xi) - 2k\pi & \text{if } \xi \in (z+, w-). \end{cases}$$

Clearly, $\varphi \in BV$ and φ is continuous at w. It is obvious that

$$|\dot{\varphi}|_{\mathcal{M}(S^1\setminus\{z\})} = |\dot{\psi}|_{\mathcal{M}(S^1\setminus\{z,w\})} = |\dot{\psi}|_{\mathcal{M}(S^1\setminus\{w\})}.$$

It follows that φ is a canonical lifting of g on $S^1 \setminus \{z\}$. Indeed, by Lemma G1 we have

$$E(g|_{S^1\setminus\{z\}}) = E(g|_{S^1\setminus\{w\}}) = |g|_{BV S^1}$$

if z, w are continuity points of g. Since

$$\begin{aligned} \varphi(z-) - \varphi(z+) &= \varphi(z-) - \varphi(w+) + \varphi(w+) - \varphi(w-) + \varphi(w-) - \varphi(z+) \\ &= \psi(w-) - \psi(w+), \end{aligned}$$

we find that the degrees obtained by cutting at z are among the ones obtained by cutting at w. By reversing the roles, we obtained that $\operatorname{Deg}_1 g$ is independent of z.

Proof of Theorem 16. Let z be a continuity point of g and let φ be a canonical lifting of g in $S^1 \setminus \{z\}$. Assume, e.g., that $z = 1$; we identify $S^1 \setminus \{1\}$ with $(0, 2\pi)$. Consider a sequence $(\varphi_n) \subset C^\infty([0, 2\pi])$ such that

$$\int_0^{2\pi} |\dot{\varphi}_n| \to |\dot{\varphi}|_{\mathcal{M}((0,2\pi))} \quad \text{and} \quad \varphi_n \to \varphi \quad \text{a.e.}$$

We may assume, in addition, that

$$\varphi_n(0) \to \varphi(0+) \quad \text{and} \quad \varphi_n(2\pi) \to \varphi(2\pi-).$$

(This is the case, e.g., if the functions φ_n are obtained from φ by mollification).

By replacing φ_n with $\varphi_n + \alpha_n x + \beta_n$, for some appropriate $\alpha_n \to 0$ and $\beta_n \to 0$, we may further assume that

$$\varphi_n(0) = \varphi(0+) \quad \text{and} \quad \varphi_n(2\pi) = \varphi(2\pi-) \quad \forall n \geq 1.$$

Set $g_n = \mathrm{e}^{i\varphi_n}$. Then, clearly,

$$g_n \in C^\infty(S^1 \setminus \{z\}) \cap C^0(S^1) \quad \text{and} \quad \deg g_n = \frac{\varphi(1-) - \varphi(1+)}{2\pi}.$$

By further mollifying g_n, we find a sequence $(h_n) \subset C^\infty(S^1; S^1)$ such that

$$h_n \to g \quad \text{a.e.}, \quad \deg h_n = \frac{\varphi(1-) - \varphi(1+)}{2\pi} \quad \forall n \geq 1,$$

and

$$\int_{S^1} |\dot{h}_n| \to \int_{S^1 \setminus \{z\}} |\dot{\varphi}| = E(g|_{S^1 \setminus \{z\}}).$$

It follows that

$$\operatorname{Deg}_2 g \supset \operatorname{Deg}_1 g.$$

Conversely, let $d \in \operatorname{Deg}_2 g$ and let $(g_n) \subset C^\infty(S^1; S^1)$ be such that

$$g_n \to g \quad \text{a.e.}, \quad \int_{S^1} |\dot{g}_n| \to |g|_{BVS^1} \quad \text{and} \quad \deg g_n = d \quad \forall n \geq 1.$$

Let $z \in S^1$ be a continuity point of g. Write, in $S^1 \setminus \{z\}$, $g_n = \mathrm{e}^{i\varphi_n}$. Then

$$\int_{S^1} |\dot{\varphi}_n| \to |g|_{BVS^1}.$$

Up to some subsequence and after subtracting a suitable multiple of 2π, we may assume that $\varphi_n \to \varphi$ a.e., where $\varphi \in BV$ is a lifting of g. Since $|\dot{\varphi}|_{\mathcal{M}(S^1 \setminus \{z\})} \leq |g|_{BVS^1}$, we find that φ has to be a canonical lifting of g. Let $\varepsilon > 0$. There is some $\delta > 0$ such that, if I is the interval of size δ centered at z, then we have $|g|_{BVS^1(I)} < \varepsilon$. We may further assume that g is continuous at the endpoints of I. Then

$$|g|_{BVS^1(S^1)} = |g|_{BVS^1(I)} + |g|_{BVS^1(S^1 \setminus I)}.$$

Arguing as above, we find that

$$\int_{S^1\setminus I} |\dot{\varphi}_n| \to |g|_{BVS^1(S^1\setminus I)} \quad \text{and} \quad \int_{I\setminus\{z\}} |\dot{\varphi}_n| \to |g|_{BVS^1(I)}.$$

In particular, for every $n \geq 1$ sufficiently large,

$$\big|(\varphi_n(z-t) - \varphi_n(z+t)) - (\varphi_n(z-) - \varphi_n(z+))\big| < 2\varepsilon \quad \forall t \in (0,\delta).$$

We pick such t so that, in addition,

$$\varphi_n(z-t) \to \varphi(z-t) \quad \text{and} \quad \varphi_n(z+t) \to \varphi(z+t).$$

We then find

$$\big|\varphi(z-t) - \varphi(z+t) - 2\pi d\big| \leq 2\varepsilon.$$

Letting $\varepsilon \to 0$ and $\delta \to 0$, we obtain

$$\varphi(z-) - \varphi(z+) = 2\pi d,$$

i.e., $d \in \mathrm{Deg}_1\, g$.

We complete the proof of Theorem 16 by proving that $g \mapsto \mathrm{Deg}\, g$ is continuous in the multivalued sense. Since Deg is $\mathbb{Z}$-valued, this amounts to proving that, for each $d \in \mathbb{Z}$, the set

$$\big\{g \in BV(S^1;S^1)\ ;\ d \in \mathrm{Deg}\, g\big\}$$

is open.

To this purpose, we start with the following

Lemma G2. *Let $g \in BV(I;S^1)$. Let $\varphi \in BV(I;\mathbb{R})$ be a lifting of g. If φ is not a canonical lifting of g, then*

$$|\dot{\varphi}|_{\mathcal{M}(I)} \geq E(g) + \pi.$$

Proof. Let A, B be the set of jump points of g, φ, respectively. Recall that

$$\begin{aligned}|\dot{\varphi}|_{\mathcal{M}(I)} &\geq |\dot{g}_d|_{\mathcal{M}(I)} + \sum_{a\in A} |\varphi(a+) - \varphi(a-)| + \sum_{a\in B\setminus A} [\varphi(a+) - \varphi(a-)| \\ &\geq E(g) + \sum_{a\in B\setminus A} |\varphi(a+) - \varphi(a-)|.\end{aligned}$$

If $B \neq A$, then

$$|\varphi(a+) - \varphi(a-)| \geq 2\pi \quad \forall a \in B\setminus A,$$

and the conclusion is clear. If $B = A$, then there is some $a \in A$ such that

$$|\varphi(a+) - \varphi(a-)| > d_{S^1}(g(a+), g(a-)),$$

for otherwise φ would be a canonical lifting. For any such a, we have

$$|\varphi(a+) - \varphi(a-)| \equiv d_{S^1}(g(a+), g(a-)) \mod 2\pi.$$

Since $d_{S^1}(g(a+), g(a-)) \le \pi$, we find that

$$|\varphi(a+) - \varphi(a-)| \ge d_{S^1}(g(a+), g(a-)) + \pi.$$

Since, for any $b \in A \setminus \{a\}$, we have

$$|\varphi(b+) - \varphi(b-)| \ge d_{S^1}(g(b+), g(b-)),$$

and the conclusion follows.

Proof of Theorem 16 completed. If $g, h \in BV(S^1; S^1)$, then

$$\text{(G7)} \qquad |g\bar{h}|_{BV} = |g\bar{h} - 1|_{BV} = |g\,(\bar{h} - \bar{g})|_{BV} \le |g|_{BV} \|h - g\|_{L^\infty} + |h - g|_{BV}.$$

Let $g \in BV(S^1; S^1)$ and let $d \in \operatorname{Deg} g$. In view of (G7), there is some $\varepsilon > 0$ such that if

$$h \in BV(S^1; S^1) \quad \text{and} \quad \|g - h\|_{BV} < \varepsilon,$$

then

$$|g\bar{h}|_{BV} < \frac{1}{10}.$$

We claim that $d \in \operatorname{Deg} h$ for any such h. Indeed, let z be a continuity point for both g and h, and let φ be a canonical lifting of g in $S^1 \setminus \{z\}$. Set $k = \bar{g}h$ and let ψ be a canonical lifting of k. Since $|k|_{BV} < \frac{1}{10}$, each jump point a of k is such that $|k(a+) - k(a-)| < \frac{1}{10}$. Thus

$$|\psi(a+) - \psi(a-)| \le 2|k(a+) - k(a-)|$$

for any such a. It follows that

$$|\psi|_{BV} = |\dot{k}_d|_{\mathcal{M}(I)} + \sum_{\substack{\text{jump points} \\ \text{of } k}} |\psi(a+) - \psi(a-)| \le 2|k|_{BV} < \frac{1}{5}.$$

Set $\phi = \varphi + \psi$. Then ϕ is a lifting of h and

$$\big|(\phi(z-) - \phi(z+)) - (\varphi(z-) - \varphi(z+))\big| < \frac{2}{5},$$

so that

$$\phi(z-) - \phi(z+) = \varphi(z-) - \varphi(z+)$$

(since both quantities are multiple of 2π). In order to complete the proof of Theorem 16, it suffices to prove that ϕ is a canonical lifting of h.

Indeed, on the one hand we have

$$E(h) \le |\dot{\phi}|_{\mathcal{M}(S^1\setminus\{z\})} \le |\dot{\varphi}|_{\mathcal{M}(S^1\setminus\{z\})} + \frac{1}{5},$$

so that $E(h) \le E(g) + \frac{1}{5}$. By reversing the roles, we obtain on the other hand that $E(g) \le E(h) + \frac{1}{5}$; thus

$$E(h) \le |\dot{\phi}|_{\mathcal{M}(S^1\setminus\{z\})} \le E(h) + \frac{2}{5}.$$

Lemma G2 implies that ϕ is a canonical lifting of h.

Proof of Theorem 17. With the notation we already used, we have

$$\big\{g\ ;\ \mathrm{Deg}\, g \text{ is single-valued}\big\} = \{g\ ;\ A_1(g) = \phi\} =: \mathcal{U}_1.$$

Thus, we have to prove that $\mathcal{U}_1$ is dense in $BV(S^1; S^1)$.

Let $g \in BV(S^1; S^1)$; then $A_1(g)$ is finite. If $A_1(g) = \phi$, then $g \in \mathcal{U}_1$. Otherwise, we may assume, for simplicity, that A_1 consists of a single point, say $A_1 = \{1\}$; the general case can be treated along the same lines. We have $|g(1-) - g(1+)| = 2$. Without loss of generality, we may assume that $g(1-) = -1$ and $g(1+) = 1$.

Given $\varepsilon > 0$, let $h_\varepsilon : S^1 \to S^1$ be given by

$$h_\varepsilon(\mathrm{e}^{i\theta}) = \begin{cases} \mathrm{e}^{i\theta} & \text{if } 1 \le \theta \le 2\pi - 1,\\ \mathrm{e}^{i(\varepsilon + (1-\varepsilon)\theta)} & \text{if } 0 \le \theta \le 1,\\ \mathrm{e}^{i((2\pi-1)(2\pi+\varepsilon) - (2\pi - 1 + \varepsilon)\theta)} & \text{if } 2\pi - 1 \le \theta < 2\pi. \end{cases}$$

It is immediate that $h_\varepsilon(1+) = \mathrm{e}^{i\varepsilon}$, $h_\varepsilon(1-) = \mathrm{e}^{-i\varepsilon}$,

$$|h_\varepsilon|_{BV} \to 0 \quad \text{and} \quad h_\varepsilon \to 1 \quad \text{uniformly.}$$

Thus,

$$gh_\varepsilon \to g \quad \text{in } BV \quad \text{as } \varepsilon \to 0.$$

On the other hand, since $h_\varepsilon \in C^0(S^1 \setminus \{1\})$, we have $A_1(gh_\varepsilon)\setminus\{1\} = A_1(g)\setminus\{1\}$. In particular, $A_1(gh_\varepsilon) \subset \{1\}$. Since, by construction, $1 \notin A_1(gh_\varepsilon)$, we have $gh_\varepsilon \in \mathcal{U}_1$. The proof of Theorem 17 is complete.

[(1)] LABORATOIRE J. -L. LIONS
UNIVERSITÉ P. ET M. CURIE, B.C. 187
4 PL. JUSSIEU
75252 PARIS CEDEX 05
E-mail address: brezis@ccr.jussieu.fr, ponce@ann.jussieu.fr

[(2)] RUTGERS UNIVERSITY
DEPT. OF MATH., HILL CENTER, BUSCH CAMPUS
110 FRELINGHUYSEN RD, PISCATAWAY, NJ 08854
E-mail address: brezis@math.rutgers.edu, augponce@math.rutgers.edu

[(3)] DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ PARIS-SUD
91405 ORSAY
E-mail address: Petru.Mironescu@math.u-psud.fr